

On projections in the noncommutative 2-torus algebra

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Abstract

We investigate a set of functional equations defining an arbitrary projection in the noncommutative 2-torus algebra \mathcal{A}_θ . The exact solutions of those provide various generalisations of the Power-Rieffel projection. By identifying the corresponding $K_0(\mathcal{A}_\theta)$ classes we get an insight into the general structure of projections in \mathcal{A}_θ .

1 Introduction

The K -theory of noncommutative 2-torus algebra \mathcal{A}_θ , known also as irrational rotation algebra, has been thoroughly investigated in the 1980's. From the works of Pimsner, Voiculescu and Rieffel (see [9, 10] and references therein) we know that its K_0 group is $\mathbb{Z} \oplus \theta\mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}$. When investigating the K -theory of a noncommutative algebra \mathcal{A} one usually has to consider projections in the matrix algebra $M_\infty(\mathcal{A})$. However, in the case of noncommutative tori it turns out that the projections in the algebra \mathcal{A}_θ itself generate the whole group $K_0(\mathcal{A}_\theta)$ (see Corollary 7.10 in [12]). Hence in this paper we

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will restrict our considerations to projections belonging to the algebra \mathcal{A}_θ itself.

The K_0 class of a projection is uniquely determined by its algebraic trace, so any two projections with the same trace must be unitarily equivalent in $M_\infty(\mathcal{A}_\theta)$ (see Corollary 2.5 in [10]). On the other hand, it has been already pointed out by Rieffel in [12] that the structure of projections in \mathcal{A}_θ is more rich than it would appear by looking at it from the K -theory level. Moreover, the projections in the algebras themselves are extensively used in modern applications, for instance in quantum physics [3, 6, 8] or theoretical engineering [7].

The aim of this paper is to shed more light on the peculiar structure of projections in the noncommutative 2-torus algebra and provide some explicit formulas which can be easily used in further applications. Below, we recall some basic definitions to fix notation and make the paper self-contained. In the next section we present functional equations defining an arbitrary projection in \mathcal{A}_θ and comment on the adopted method of solving those. In section 3 we investigate some very special solutions - the Power-Rieffel type projections. Those will serve us firstly to provide an alternative proof of the important Corollary 7.10 from [12]. Secondly, we will use them as a starting point for generalisations to come in section 4. Finally, in section 5, we make conclusions and discuss some open question that arose during the presented analysis.

The algebra of noncommutative 2-torus is an universal C^* -algebra generated by two unitaries U, V satisfying the following commutation relation

$$UU^* = VV^* = 1 \quad VU = e^{2\pi i \theta} UV$$

for some real parameter $\theta \in [0, 1)$, which we assume to be irrational. Since \mathcal{A}_θ is a C^* -algebra one can regard its elements as continuous functions of the generators [5, 11]. The elements we will be using in this paper may be written as finite sums

$$\mathcal{A}_\theta \ni a = \sum_{m=-M}^M U^m a_m(V), \quad a_m(y) \in C(\mathbb{R}/\mathbb{Z}), \quad (1)$$

thus they actually belong to a dense $*$ -subalgebra of \mathcal{A}_θ [11].

The noncommutative 2-torus algebra is equipped with a canonical trace defined by

$$\tau(a) = \int_0^1 a_0(x) \, dx.$$

We shall use it to determine the K_0 class of a projection on the strength of Corollary 2.5 in [10]. For more details of the noncommutative torus structure the reader may refer to [2, 5, 13].

2 Equations for general projections in \mathcal{A}_θ

Having recalled the basic features of the noncommutative 2-torus algebra we are ready to investigate the structure of projections in it.

We start with providing a set of functional equations a projection must satisfy. Let us consider an element of \mathcal{A}_θ written in the form (1)

$$p = \sum_{n=-M}^M U^m p_n(V), \quad \text{for some } M \in \mathbb{N}. \quad (2)$$

The conditions for p to be a projection yield a set of functional equations for the functions $p_i(x) \in C(\mathbb{R}/\mathbb{Z})$

$$p_k(x) = \overline{p_{-k}}(x + k\theta), \quad \text{for } k = -M, \dots, M, \quad (3)$$

$$p_k(x) = \sum_{m,a=-M}^M p_m(x + a\theta) p_a(x) \delta_{m+a,k}, \quad \text{for } k = -M, \dots, M, \quad (4)$$

$$0 = \sum_{m,a=-M}^M p_m(x + a\theta) p_a(x) \delta_{m+a,k}, \quad \text{for } k < -M \text{ and } k > M. \quad (5)$$

Some of the above equations are redundant and the number of independent ones is $3M + 2$. It can be easily seen by noticing that the equations (3-5) with $k < 0$ are equivalent to those with $k > 0$, because the functions $p_k(x)$ with negative indices are actually defined by (3) with $k > 0$. For $M = 0$ formulas (3-5) imply $p_0(x) \equiv 1$ as one may expect. When $M = 1$ one obtains the familiar Power-Rieffel equations [11]. However, for $M \geq 2$ the equations become more and more involved and even the existence of a solution is not

obvious. In [3] we found four particular solutions to (3-5) with $M = 2$, which represent different classes of $K_0(\mathcal{A}_\theta)$. In the next sections we present a generalisation of the construction given in [3]. Before we start solving the equations (3-5) let us adopt the following definition and make some general comments.

Definition 2.1. *We say a projection is of order M if it is of the form (2) and $p_M(x) \neq 0$.*

We will consider only real-valued functions although (3) requires only $p_0(x)$ to be real. Moreover, we have already noted that (3) defines the functions $p_k(x)$ for $k < 0$ and it is convenient to get rid of the functions $p_k(x)$ with negative index k in the equations (4) and (5) before solving them. We shall present special solutions such that each summand on the RHS of (5) is equal to zero independently. The same should hold for summands of (4) with $k > 0$ excluding those with $m = 0$ or $a = 0$ – these are combined to form equations

$$p_k(x)(p_0(x) + p_0(x + k\theta) - 1) = 0, \quad (6)$$

which we also require to be satisfied independently. The equations (4) with $k < 0$ are redundant and the case $k = 0$ cannot be split into independent equations. After (3) is substituted into (4) for $k = 0$ we obtain

$$\begin{aligned} p_M^2(x - M\theta) + p_M^2(x) + p_{M-1}^2(x - (M-1)\theta) + p_{M-1}^2(x) + \\ + \dots + p_1^2(x - \theta) + p_1^2(x) + p_0(x)(p_0(x) - 1) = 0. \end{aligned} \quad (7)$$

Although the imposed additional properties seem to be extremely restrictive, the obtained structure of solutions is rich enough to build many interesting projections. The general formula for solutions to (3-5) with a given M and trace $n\theta$ is complicated enough to obscure the underlying structure, so we shall rather present the construction step by step.

3 Power-Rieffel type projections

We shall begin with the simplest example of a projection, namely let us set $p_k(x) \equiv 0$ for all $1 \leq k \leq M - 1$. Then (3-5) reduce to the Power-Rieffel

equations with parameter $M\theta$

$$p_M(x + M\theta)p_M(x) = 0, \quad (8)$$

$$p_M^2(x) + p_M^2(x - M\theta) + p_0(x)(p_0(x) - 1) = 0, \quad (9)$$

$$p_M(x)(1 - p_0(x) - p_0(x + M\theta)) = 0. \quad (10)$$

The solutions to (8-10) are known as Power-Rieffel type projections [4, 6]

$$p_0(x) = \begin{cases} d_M(x), & 0 \leq x \leq \varepsilon_M \\ 1, & \varepsilon_M < x < M\theta \\ 1 - d_M(x - M\theta), & M\theta \leq x \leq M\theta + \varepsilon_M \\ 0, & M\theta + \varepsilon_M < x < 1 \end{cases}, \quad (11)$$

$$p_M(x) = \begin{cases} \sqrt{d_M(x)(1 - d_M(x))}, & 0 \leq x \leq \varepsilon_M \\ 0, & \varepsilon_M < x \leq 1 \end{cases}, \quad (12)$$

where $d_M(x)$ is a continuous function with $d_M(0) = 0$ and $d_M(\varepsilon_M) = 1$. An example of such projection is depicted in figure 1.

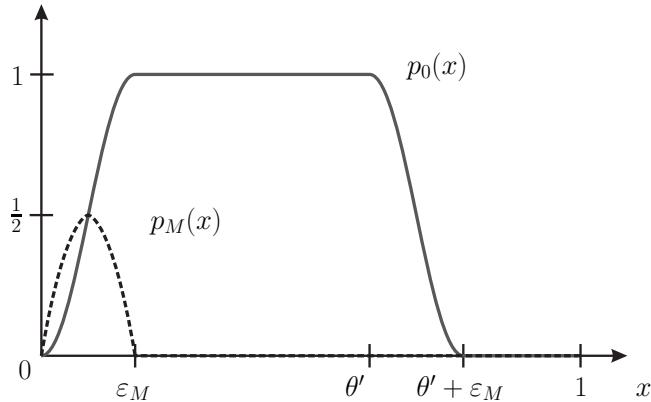


Figure 1: Depiction of a Power-Rieffel type projection. θ' stands for the fractional part of $M\theta$ and we have $0 < \varepsilon_M < \theta'$, $\varepsilon_M + \theta' < 1$.

As an almost immediate consequence of the existence of such projections in \mathcal{A}_θ we get the following property of the noncommutative 2-torus.

Theorem 1. *The algebra \mathcal{A}_θ contains projections representing infinitely many different classes of $K_0(\mathcal{A}_\theta)$.*

Proof. First let us recall, that the algebraic trace of a projection must take a value in $[0, 1] \subset \mathbb{R}$ with the end-points reached by trivial-projections. Now, due to the periodicity of $p_i(x)$ functions the equations (8-10) are invariant with respect to the transformation $M\theta \rightarrow M\theta + z$ for any $z \in \mathbb{Z}$. This means that a Power-Rieffel type projection of order M has the algebraic trace equal to $M\theta - n$, where n denotes the biggest integer smaller than $M\theta$. Since θ is irrational we have infinitely many M such that

$$0 < M\theta - n < 1 \iff \frac{n}{M} < \theta < \frac{n+1}{M}.$$

□

One may notice that the above theorem is actually a consequence of the Corollary 7.10 in [12] for the case of two-dimensional noncommutative torus. We present it here, since it arose naturally in the investigation of the symmetry of equations (8-10). Moreover, in our method of proving Theorem 1 we have identified the concrete projections representing a given K_0 class, whereas in [12] only the existence of those has been stated. In the next section we shall use the Power-Rieffel type projections as a starting point for various generalisations.

4 General projections in \mathcal{A}_θ

Let us now see what kind of projections we can get by letting functions $p_k(x)$ in (2) to be non-zero for some of the indices $k \in \{1, \dots, M-1\}$. The results are stated in the following propositions.

Proposition 2. *A projection of order M may represent the $K_0(\mathcal{A}_\theta)$ class $[n\theta]$, as well as the class $[1 - n\theta]$, for all $n = 1, 2, \dots, \frac{1}{2}M(M+1)$, provided that $0 < \theta < 1/\max(n, M)$.*

By $[n\theta] \in K_0(\mathcal{A}_\theta)$ we denote the K_0 class represented by a projection $p \in \mathcal{A}_\theta$ with $\tau(p) = n\theta$.

Proposition 3. *The equations (3-5) for a projection of order M admit solutions with $p_k(x) \neq 0$ for every $k \in \{0, \dots, M\}$ whenever $0 < \theta < 1/M$.*

We shall start with the proof of Proposition 2 by showing how to use the $p_k(x)$ functions to increase or decrease the trace of a Power-Rieffel type projection. Then we present a method of including the remaining $p_k(x)$ functions to the projections constructed in the previous proof without changing its traces. In this way we will prove Proposition 3. Both proofs are constructive so we are able to plot some examples of the $p_0(x)$ functions of the relevant projections which, as we shall see, determine uniquely all of the other functions $p_k(x)$ for $k \neq 0$. A brief discussion of the assumptions limiting the θ parameter may be found in section 5.

Proof of Proposition 2. Let us start with the case of $\tau(p) = n\theta > M\theta$. We shall begin with a Power-Rieffel type projection as defined in the previous section (11, 12). First note that if $M\theta < 1$ then the functions $p_0(x)$ and $p_M(x)$ of Power-Rieffel type projection vanish for $x \geq M\theta + \varepsilon_M$. If θ is small enough (*i.e.* $(M+k)\theta < 1$) then we can “glue” a Power-Rieffel type projection of trace $k\theta$ to the previous one. Namely, let us define

$$p_0(x) = \begin{cases} d_k(x), & M\theta + \varepsilon_M \leq x \leq M\theta + \varepsilon_M + \varepsilon_k \\ 1, & M\theta + \varepsilon_M + \varepsilon_k < x < (M+k)\theta + \varepsilon_M \\ 1 - d_k(x - k\theta), & (M+k)\theta + \varepsilon_M \leq x \leq (M+k)\theta + \varepsilon_M + \varepsilon_k \\ 0, & (M+k)\theta + \varepsilon_M + \varepsilon_k < x < 1 \end{cases},$$

$$p_k(x) = \begin{cases} \sqrt{d_k(x)(1 - d_k(x))}, & M\theta + \varepsilon_M \leq x \leq M\theta + \varepsilon_M + \varepsilon_k \\ 0, & \text{elsewhere} \end{cases},$$

for a smooth function $d_k(x)$ with $d_k(M\theta + \varepsilon_M) = 0$, $d_k(M\theta + \varepsilon_M + \varepsilon_k) = 1$ and a small parameter ε_k . The summands of (4) and (5), which we have assumed to be equal to zero independently, have the form $p_m(x + a\theta)p_a(x)$. This means that all of the non-zero functions $p_k(x)$ for $k \neq 0$ shifted to the interval $x \in [0, \theta]$ must not intersect. The latter can be fulfilled by restricting the ε parameters

$$\varepsilon_M + \varepsilon_k < \theta, \quad \varepsilon_M + \varepsilon_k + (M+k)\theta < 1. \quad (13)$$

The imposed restrictions on ε parameters imply that equation (7) reduces to two equations of the form (9). Namely for $x \in [0, \varepsilon_M] \cup [M\theta, M\theta + \varepsilon_M]$ and

for $x \in [M\theta + \varepsilon_M, M\theta + \varepsilon_M + \varepsilon_k] \cup [(M+k)\theta + \varepsilon_M, (M+k)\theta + \varepsilon_M + \varepsilon_k]$ we have respectively

$$\begin{aligned} p_0(x)(1 - p_0(x)) &= p_M^2(x) + p_M^2(x - M\theta), \\ p_0(x)(1 - p_0(x)) &= p_k^2(x) + p_k^2(x - k\theta). \end{aligned}$$

These equations are satisfied by construction of $p_k(x)$ and $p_M(x)$. On the remaining part of the interval $[0, 1]$ the equation (7) is trivially satisfied, since both LHS and RHS are equal to 0. By the same argument equation (6) remains satisfied since it is satisfied for both Power-Rieffel type projections independently. Thus we have obtained a new projection with a trace $(M + k)\theta$. Examples of such projections are depicted in figure 2.

If the parameter θ is small enough (*i.e.* $n\theta < 1$) we can continue the process of “gluing” Power-Rieffel type projections to obtain a projection bearing the trace $n\theta$. If one makes use of all the functions $p_k(x)$ for $1 \leq k \leq M-1$ to increase the trace one will end with a projection bearing the trace $(1+2+\dots+M)\theta = \frac{1}{2}M(M+1)\theta$, in agreement with Proposition 2. The only thing one has to take care of are the conditions satisfied by the parameters ε_k . The restrictions (13) may be easily generalized to the case of $1 \leq s \leq M-1$ non-vanishing $p_{k_s}(x)$ functions

$$\varepsilon_{k_1} + \dots + \varepsilon_{k_s} + \varepsilon_M < \theta, \quad \varepsilon_{k_1} + \dots + \varepsilon_{k_s} + \varepsilon_M + n\theta < 1. \quad (14)$$

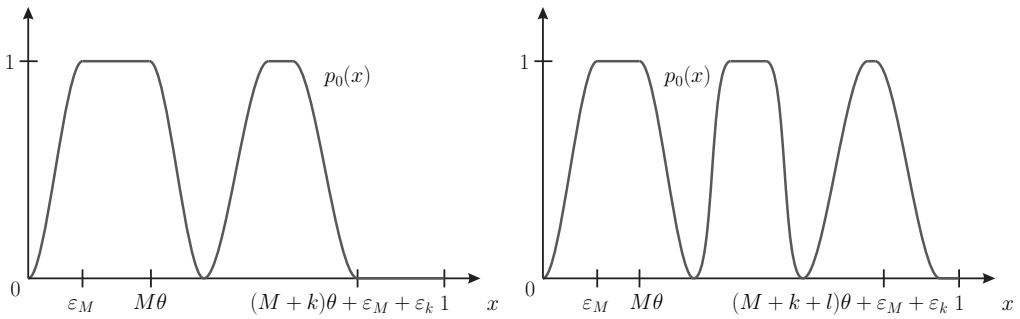


Figure 2: Examples of projections with traces $(M + k)\theta$ and $(M + k + l)\theta$.

Let us now consider the case of projections of order M and trace $n\theta$ with $1 \leq n < M$. Again we shall use as a starting point a Power-Rieffel type

projection (11, 12), but now we will “cut out” a part of it. Let us set

$$p_0(x) = \begin{cases} d_k(x), & \varepsilon_M \leq x \leq \varepsilon_M + \varepsilon_k \\ 0, & \varepsilon_M + \varepsilon_k < x < k\theta + \varepsilon_M \\ 1 - d_k(x - k\theta), & k\theta + \varepsilon_M \leq x \leq k\theta + \varepsilon_M + \varepsilon_k \\ 1, & k\theta + \varepsilon_M + \varepsilon_k < x < M\theta \end{cases},$$

$$p_k(x) = \begin{cases} \sqrt{d_k(x)(1 - d_k(x))}, & \varepsilon_M \leq x \leq \varepsilon_M + \varepsilon_k \\ 0, & \text{elsewhere} \end{cases},$$

with a continuous function $d_k(x)$ such that $d_k(\varepsilon_M) = 1$, $d_k(\varepsilon_M + \varepsilon_k) = 0$. The conditions $\varepsilon_M + \varepsilon_k < \theta$ and $\varepsilon_M + M\theta < 1$ should be satisfied. The situation now is completely analogous to the case of “glued” projections and the same arguments apply. A projection obtained in this way bears the trace $(M - k)\theta$ for $1 \leq k \leq M - 1$ (see figure 3).

To end the proof of Proposition 2 it remains just to recall that if p is a projection then obviously $1 - p$ is so. This means that all of the considerations hold for projections of traces $(1 - n\theta)$ - one simply should take $1 - p_0(x)$ instead of $p_0(x)$ and leave $p_k(x)$ for $k \neq 0$ as they are. \square

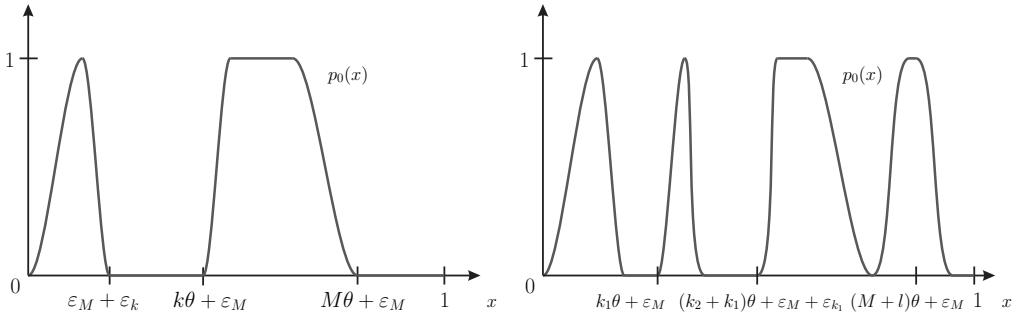


Figure 3: Examples of projections of trace $(M - k)\theta$ and $(M - k_1 - k_2 + l)\theta$.

The presented proof provides a great variety of possible projections with a given trace, which have, in general, different orders. Let us notice that the two procedures of increasing and decreasing the trace of a projection of a given order can be applied simultaneously and in arbitrary sequence (see figure 3). One only has to choose well the parameters ε_k to have the equations (14) satisfied. The equations (14) guarantee that the functions $d_k(x)$

do not superpose and the equations (3-5) remain satisfied. This leads us to an enormous number of projections if the order N is big enough. Let us now pass on to the most general projections we have found.

Proof of Proposition 3. One can let all of the $p_k(x)$ function to be non-zero by incorporating to $p_0(x)$ some “bump functions” $d_k(x)$. As a starting point one should take an arbitrary projection defined in section 3 or 4. For sake of simplicity let us now denote by k the free indices, *i.e.* those that has not been used for the construction of the starting point projection. Now, if one sets $p_0(x) = d_k(x)$ for $x \in [\delta_k, \delta_k + \varepsilon_k]$, with $d_k(\delta_k) = d_k(\delta_k + \varepsilon_k) = 1$ or $d_k(\delta_k) = d_k(\delta_k + \varepsilon_k) = 0$ then (6) forces us to set $p_0(x) = 1 - d_k(x - k\theta)$ for $x \in [k\theta + \delta, k\theta + \delta + \varepsilon_k]$. The function $p_k(x)$ should be then defined as previously by $\sqrt{d_k(x)(1 - d_k(x))}$ for $x \in [\delta_k, \delta_k + \varepsilon_k]$ and 0 elsewhere, so that (7) remains fulfilled. The only task to accomplish is to choose well the parameters ε_k and δ_k to avoid the possible intersection of $d_k(x)$ functions. The parameters ε_k should be such that the equations (14) remain satisfied, and δ_k would then take a form $n\theta + \varepsilon_{k_1} + \dots + \varepsilon_{k_s}$ for $n, s \in \mathbb{Z}$ which depend on the concrete projection one has chosen as a starting point. \square

Examples of the described projections are shown in figure 4.

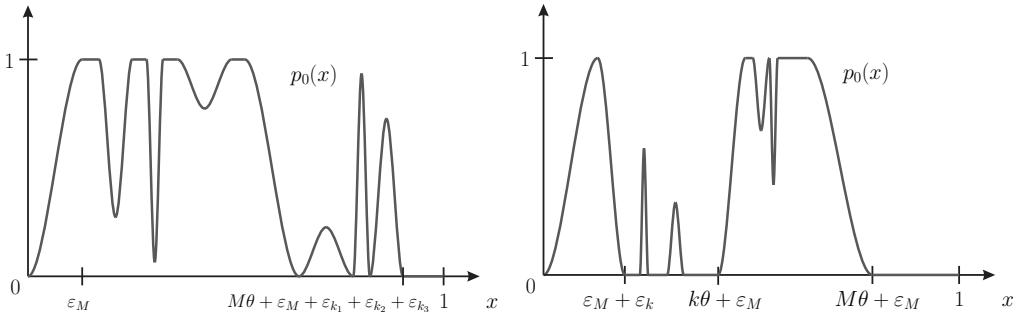


Figure 4: Examples of projections of traces $M\theta$ and $(M - k)\theta$.

By giving constructive proofs of Propositions 2 and 3 we have exhausted all of the possibilities of constructing projections in \mathcal{A}_θ with the method described in section 2. Let us now pass on the last section to summarise the obtained results and outline the directions of possible further investigations.

5 Final remarks

In the previous section we have presented many projections, which are true generalisations of the Power-Rieffel projection. Some represented the same $K_0(\mathcal{A}_\theta)$ class, but had different orders. The others conversely - had the same order, but different traces. A natural question one can ask is what are the relations between the presented projections? The answer is provided by Theorem 8.13 in [12]. It tells that if two projections in \mathcal{A}_θ represent the same $K_0(\mathcal{A}_\theta)$ class (hence have the same trace), then not only they are unitarily equivalent in $M_\infty(\mathcal{A}_\theta)$, but they are actually in the same path component of the set of projections in \mathcal{A}_θ itself. This means that there exists a homotopy of projections in \mathcal{A}_θ for any two of projections in \mathcal{A}_θ which have the same trace. For instance, if one takes $d_k(t, x) := td_k(x) + (1 - t)$ instead of $d_k(x)$ with $d_k(\delta_k) = d_k(\delta_k + \varepsilon_k) = 1$ in the projections constructed at the end of preceding section, then one would obtain a projection for all $t \in [0, 1]$.

In consequence, from the topological point of view it is sufficient to consider Power-Rieffel type projection, since they are the generators of the $K_0(\mathcal{A}_\theta)$ group. On the other hand, the richness of the projection structure may show up in applications, at the level of model building, where the exact formulas for projections are needed [6, 7]. Let us also mention, that we were able to derive the new projections in \mathcal{A}_θ by purely analytic methods without using a concrete realisation (as in [7] or [12]) nor referring to the theory of special functions (as in [1]).

The puzzling thing about the newly found projections is that their existence in \mathcal{A}_θ seems to depend on the noncommutativity parameter θ as stated in the propositions in section 4. Unfortunately the solution presented there cannot be adapted to the case $n\theta > 1$, as it was done for the Power-Rieffel type projections in section 3. It is so because the symmetry of (8-10), used in the proof of Theorem 1 is absent in general equations (3-5). Note that the discussed symmetry is also broken whenever we introduce the mentioned “bump functions”. Whether there is a real difference in the structure of projections in \mathcal{A}_θ depending on the noncommutativity parameter θ or is it just an artefact of our method of solving (3-5) remains an open question.

Let us end this paper with a remark that the presented procedure of

constructing projections may be generalised to higher dimensional tori. This would provide explicit formulas for the generators of $K_0(C(\mathbb{T}_\theta^n))$ and give more insight into the geometry of noncommutative spaces.

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